INTERACTION OF IONS WITH A SPECTRUM OF ELECTROSTATIC WAVES

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Abstract: A *single* electrostatic, lower-hybrid wave, propagating across a magnetic field,can energize ions only if the ion dynamics becomes chaotic [1]. We show that low energy ions, which would not gain energy from a single wave, can gain energy from two or more waves provided the wavelengths and frequencies are chosen appropriately. An appropriate choice can also lead to an extraction of ion energy by the waves.

In this paper we give results of our analytical and numerical studies on the dynamics of ions when interacting with two electrostatic waves in the lower-hybrid frequency range. We find that, for appropriate choices of the wave parameters (amplitudes, frequencies, and wavelengths), the dynamical phase space cannot be distinctly divided as in the case of one electrostatic wave. We analytically deduce, and numerically observe, a new phenomenon of nonlinear, coherent energization by which ions, with initially low energies, started off in the coherent part of phase space can be energized into the chaotic phase space. Thus, low-energy ions, which would be unaffected when interacting with a single wave, can be energized by two (or more) waves. In particular, for two waves, the wave frequencies have to be separated by an integer (≤ 3) multiple of the ion cyclotron frequency, and the higher frequency wave should have the shorter wavelength. For an appropriate choice of the wave parameters, the waves can also be used to remove energy from the ions by this nonlinear process. Since the frequency and wavelength spectra of lower-hybrid waves can be externally controlled in tokamak experiments, it should be possible to either accelerate or decelerate ions of a particular species, and from a predetermined region of phase space. Experiments can be set up to test these nonlinear, coherent energization or de-energization of ions.

The motion of an ion, of charge Q and mass M, interacting with two plane electrostatic waves, propagating perpendicularly (along $\hat{x}$) to an ambient, uniform, magnetic field $\vec{B} = B_0\hat{z}$, is given by:

$$\frac{d^2x}{d\tau^2} \;+\; x \;=\; \epsilon_1 \sin\left(x - \nu_1\tau\right) \;+\; \epsilon_2 \sin\left(\kappa x - \nu_2\tau\right) \tag{1}$$

where x is the normalized (to the wavelength of the first wave $2\pi/k_1$) position, τ is the time normalized to the inverse of the ion-cyclotron frequency $\Omega = QB_0/M$, ν's are the wave frequencies normalized to Ω, $\kappa = k_2/k_1$, $\epsilon_i = QE_ik_1/(M\Omega^2)$, and E_i is the electric field. A parameter which is useful for describing the ion dynamics is the normalized ion Larmor radius $\rho = \sqrt{x^2+\dot{x}^2}$ where $\dot{x} = dx/d\tau$.

In general, it has been found that an ion gains energy from a single wave only if its motion becomes chaotic [1]. The ion motion becomes chaotic for $\epsilon \;>\; \epsilon_{th} \;\approx \nu^{2/3}/4$ provided the initial $\rho = \rho_0$ satisfies: $\nu - \sqrt{\epsilon} \lesssim \rho_0 \lesssim \;(2/\pi)^{1/3}\,(4\epsilon\nu)^{2/3}$ The left-hand side of the above inequality gives the lower bound, in ρ, of the chaotic phase space.

For ions whose energies are below the lower energy bound of the chaotic region, we expect that their dynamics can be determined analytically. Towards that end, we carry out a perturbation analysis of Eq. (1) using the method of multiple time scales [2]. A more comprehensive and general analytical treatment using the Lie transform perturbation technique has also been developed and is presented elsewheres [3]. The perturbation parameter, in the method of multiple time scales, is the normalized amplitude of the waves. In our analysis we assume that neither ν_1 nor ν_2 is an integer, i.e. the wave frequencies are not an integer multiple of the ion-cyclotron frequency. However, we will assume that the difference in the frequencies of the two waves is an integer multiple of the ion-cyclotron frequency, i.e. $\nu_1 - \nu_2 = N$, an integer. (The analysis can be generalized to the case when $\nu_1 + \nu_2 = N$, and also for ν_1 and ν_2 are integers [3].) Our analysis breaks down in the vicinity of the chaotic regime. Upon carrying the multiple time scale analysis to second order in the amplitudes, we find that an approximate solution of (2) is given by:

$$x(\tau) \;\approx\; \rho(\tau)\sin\left\{\tau + \psi(\tau)\right\}. \tag{2}$$

The evolution equations for $\rho(\tau)$ and $\psi(\tau)$ are:

$$\frac{\partial\rho}{\partial\tau} \;\;=\;\; -\frac{\epsilon_1\epsilon_2}{2\rho}N\sin(N\psi)\sum_{l=-\infty}^{\infty}\frac{\mathrm{J}_l\left(\rho\right)\mathrm{J}_{l-N}\left(\kappa\rho\right)}{1-(l-\nu_1)^2} \tag{3}$$

$$\frac{\partial \psi}{\partial \tau} = -\frac{1}{4\rho}\frac{\partial}{\partial \rho}\sum_{l=-\infty}^{\infty}\frac{\epsilon_1^2 \mathrm{J}_l^2(\rho) + \epsilon_2^2 \mathrm{J}_{l-N}^2(\kappa\rho)}{1-(l-\nu_1)^2} - \frac{\epsilon_1\epsilon_2}{2\rho}\frac{\partial}{\partial \rho}\sum_{l=-\infty}^{\infty}\frac{\mathrm{J}_l(\rho)\,\mathrm{J}_{l-N}(\kappa\rho)}{1-(l-\nu_1)^2} \tag{4}$$

If $\epsilon_1 = \epsilon_2 = \epsilon$, then the above equations become independent of amplitude if we define a new time variable $\overline{\tau} = \epsilon^2\tau$. This implies that the change in the Larmor radius of an ion is independent of the amplitude of the two waves. However, the rate at which the Larmor radius changes is inversely proportional to the square of the amplitude. This is an important result. Even if the wave amplitudes are below the threshold for onset of chaotic motion, ions can still get energized. This would not occur in the case of a single wave.

Upon substituting $I = \rho^2/2$, the above evolutions equations for the amplitude and the phase can be derived from the Hamiltonian:

$$\overline{H}(I,\psi) = S_1(I) + \cos(N\psi)\,S_2(I) \tag{5}$$

where S_1 and S_2 are the following sums:

$$S_1(I) = \frac{1}{4}\sum_{l=-\infty}^{\infty}\frac{\epsilon_1^2 \mathrm{J}_l^2(\rho) + \epsilon_2^2 \mathrm{J}_{l-N}^2(\kappa\rho)}{(l-\nu_1)^2-1}, \quad S_2(I) = \frac{1}{2}\,\epsilon_1\epsilon_2\sum_{l=-\infty}^{\infty}\frac{\mathrm{J}_l(\rho)\,\mathrm{J}_{l-N}(\kappa\rho)}{(l-\nu_1)^2-1} \tag{6}$$

The Hamiltonian (5) is independent of time signifying that it is a new constant, or invariant, (to second order in the amplitude) of the dynamics. The ion orbits obtained by this Hamiltonian are in very good agreement with the results obtained from the numerical integration of (1).

In the figure we show the time evolution of ρ for two ions that are started at different energies. The waves are assumed to have the same wavelengths, but their frequencies differ by the ion-cyclotron frequency. The two ions undergo nonlinear, coherent energization. The ion started at the higher initial energy makes it into the chaotic phase space where it undergoes diffusive motion. The dynamics of the nonlinear, coherent energization are completely explained by Eqs. (5)-(6). From these equations it is simple to understand why one of the ions makes it into the chaotic region while the other one does not.

This result has been extended to a broad spectrum of waves. We find that the second-order perturbation analysis applies to the case of multiple waves and that, if there are enough pairs of waves which are separated by an integer multiple of the ion-cyclotron frequency, the mechanism of nonlinear, coherent energization persists.

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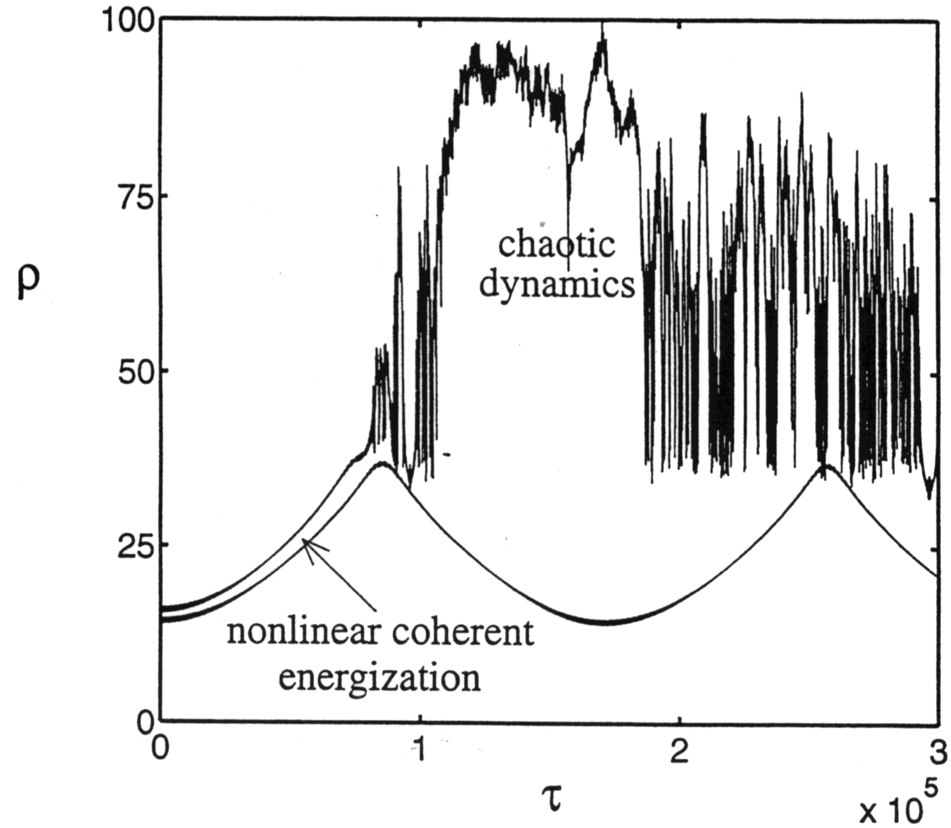

ρ verus τ for two deuterium ions which initially have $\rho_0 = 14.8$ and 16.3. The other parameters are $\epsilon_1 = \epsilon_2 = 4.9$, $\nu_1 = 40.3$, $\nu_2 = 41.3$, and $\kappa = 1$.

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